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The Consistency Argument for Ranking Functions

Abstract. The paper provides an argument for the thesis that an agent's degrees of disbelief should obey the ranking calculus. This Consistency Argument is based on the Consistency Theorem. The latter says that an agent's belief set is and will always be consistent and deductively closed iff her degrees of entrenchment satisfy the ranking axioms and are updated according to the ranktheoretic update rules.

Keywords: Conditionalization, Conditional Consistency, Consistency, Consistency Argument, Consistency Theorem, Deductive Closure, Dutch Book Argument, Ranking Functions, Probability Measures, Revision, Spohn, Update Rule.

1. Introduction

In his (1998) James Joyce provides an *epistemic* vindication of the thesis that an agent's degrees of belief should obey the probability calculus in the sense that her degree of belief function be non-negative, normalized, and finitely additive. Rather than the supposedly pragmatic vindication provided by a Dutch Book Argument, he aims at a genuine non-pragmatic vindication of probabilism. Joyce's argument is based on the assumption that an agent's degree of belief function is epistemically defective if there exists another degree of belief function which is more *accurate* in each possible world. Accuracy of an agent's degree of belief in a proposition A at some possible world ω is identified with the *distance* between the agent's degree of belief in A and the truth value of A in ω (1 for true, 0 for false).

Apart from some more technical objections pertaining to the way distance is measured (Maher 2002), there are the following alleged problems with Joyce's argument. For one, Joyce's conditions on measures of inaccuracy do not determine a single measure, but a whole set of such measures. This in itself would rather strengthen than weaken Joyce's argument, were it not for the fact that these measures differ in their recommendations as to which alternative degree of belief function an incoherent degree of belief function should be replaced by. All measures of inaccuracy agree that an incoherent agent whose degree of belief function violates the probability axioms should adopt another coherent degree of belief function which is more

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accurate in each possible world. However, these measures may differ in their recommendation as to which particular coherent degree of belief function the agent should adopt. In fact, for each possible world, following the recommendation of one measure will leave the agent off less accurate according to some other measure. *Bronfman's objection* questions the normative force of Joyce's argument on the basis of this fact. Why should I move from my incoherent degree of belief function to a coherent one, if my advisors, though agreeing *that* I should move, all disagree as to *where* I should move. In Joyce's terms (slightly adapted), suppose for each American city there is a more beautiful Australian city. I live in Pasadena, and according to any reasonable index of beauty there is some Australian city which is more beautiful than Pasadena according to this index. The big city close to the sea index says I should move to Melbourne or Sydney (but not to Canberra). The clean air index recommends moving to Canberra (but not to Melbourne or Sydney). And so on. Why should I move to Australia when all these standards disagree on where in Australia I should go, even if they all agree I should go somewhere in Australia? See Bronfman (manuscript).

Then there is the intuition that it is better to be accurate in the actual world, possibly at the cost of being incoherent and inaccurate on average, than to be coherent and accurate on average, but inaccurate in the actual world. *Hájek's objection* says that a rational agent will prefer to have a low rather than high degree of belief in the proposition that it is safe to jump from the Eiffel Tower, even if this comes at the cost of having an incoherent degree of belief function. See Hájek (to appear).

Moreover, it turns out that Joyce's theorem depends on representing 'true' by 1 and 'false' by 0, and is false if 'true' is represented by 0 and 'false' by 1. *Howson's objection* is that Joyce's theorem has no epistemic significance insofar as it is not a result about truth, but only a result about distributions ω of 1s and 0s over propositions A, B in some field which satisfy $\omega(A) \in \{0,1\}$, $\omega(\overline{A}) = 1-\omega(A)$, and $\omega(A \cap B) = \omega(A) \cdot \omega(B)$. See Howson (manuscript).

I am not convinced by these objections. Bronfman's objection is nothing but a plea to further narrow down the class of inaccuracy measures. Suppose with Andy Egan (slightly adapted) that all moral theories agree that rich countries should help poor countries, but they all disagree as to how to help. One moral theory says rich countries should ship food to poor countries rather than importing food from them (in order for them to have enough food). Another moral theory says rich countries should import food from poor countries rather than shipping food to them (in order to strengthen their economy). That does not imply that rich countries need not help poor countries after all.

Hájek's objection neglects that the counterintuitive character of the above example depends on our low degree of belief that jumping from the Eiffel Tower is safe. It is true that the actual world has a privileged status. But we do not know which world is actual. All we have are our degrees of belief. If you are confident that jumping from the Eiffel Tower is not safe, your *expected* inaccuracy will take this bias into account. The expected inaccuracy (in the sense of an inaccuracy measure I) of a degree of belief function B is given by $\sum_{\omega \in W} I(B, \omega) \cdot Bel(\omega)$. Bel is your *actual* degree of belief function on the finite set of possibilities W, and $I(B, \omega)$ is the inaccuracy of B in the world ω. Joyce considers the inaccuracy $I(B, \omega)$ in each world $\omega \in W$ and so assumes all worlds to be on a par as far as inaccuracy is concerned. Trivially, Joyce's theorem remains a theorem if '*expected* inaccuracy' is substituted for 'inaccuracy in each possible world', provided the actual belief function Bel assigns a positive degree of belief to at least one world. So although Joyce's original argument may be subject to Hájek's objection, the substitute in terms of expected inaccuracy is not.

Howson's objection finally locates the assumption of representing 'true' by 1 and 'false' by 0 in Joyce's conditions on measures of inaccuracy. But it is the probability axioms themselves that adopt this assumption. This is seen in the normalization axiom that requires the probability for the whole set of possibilities (the tautology) to be 1. Joyce's theorem is true for the bundle consisting of the probability axioms and the labelling convention to represent 'true' by 1 and 'false' by 0. It is also true for a dual of the probability axioms ($\Pr(A) \leq 1$, $\Pr(W) = 0$, and $\Pr(A \cap B) = \Pr(A) + \Pr(B)$ if $A \cup B = W$) and the dual convention of representing 'true' by 0 and 'false' by 1.[1]

But why all this ado about accuracy when there already exists a vindication of probabilism? The desire for an epistemic vindication seems to have arisen out of the shortcomings of the pragmatic nature of the Dutch Book Argument and the apparently not completely successful efforts to depragmatize the latter (Armendt 1993, Christensen 1996, Howson and Franklin 1994, Skyrms 1984, Ramsey 1926). On my preferred reading, these depragmatiza-

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[1] According to Colin Howson (personal correspondence), the probability axioms make no assumption concerning the numerical representation of truth values, because they do not mention truth or falsity. They only mention logical truth and logical falsity, and the numbers 1 and 0 assigned these are interpretable in ways that have nothing to do with truth or falsity (e.g. as infinite odds).

On a slightly different note, I do not know of a way to fix things if 'true' is represented by $+\infty$ and 'false' by $-\infty$.

tion efforts are based on a distinction between degrees of belief on the one hand and fair betting ratios on the other, and the idea that degrees of belief for propositions are (measured by) evaluations of fair betting ratios for these propositions. Violating the probability axioms then does not only result in the pragmatic defect of being vulnerable to a sure loss. First and foremost it is the epistemic defect of being inconsistent in one's evaluations of fair betting ratios.

Yet inconsistent evaluations of fair betting ratios for various propositions, while being inconsistencies, are not inconsistent beliefs *in those propositions*. Indeed, unless degrees of belief give rise to qualitative yes-or-no beliefs, it is a category mistake to say that incoherent degrees of belief for propositions amount to inconsistent beliefs in those propositions. Inconsistency is only well defined for propositions or sentences[2]. Coherent degrees of belief do not give rise to consistent and deductively closed beliefs, at least if belief is degree of belief to some degree. The impossibility result in question is, of course, the well known lottery paradox (Kyburg 1961, Hempel 1962).

Given this background *the* epistemic vindication of a normative theory of epistemic states, such as subjective probability theory, seems to be something along the following lines. An agent's epistemic states should obey such and such axioms, because the set of her beliefs based on these epistemic states is consistent and deductively closed just in case her epistemic states satisfy the axioms in questions. The requirements of consistency and deductive closure can already be found in Hintikka (1962), and have become the defining properties of a belief set. Obviously, for such a justification to work, epistemic states and the axioms governing them must give rise to a notion of belief. But not only subjective probabilities do not do this. The same holds true for Dempster-Shafer belief functions (Dempster 1968, Shafer 1976) as well as plausibility measures (Halpern 2003).[3]

2. Ranking Functions

Hence, the first question is whether there is a representation of epistemic states that gives rise to a notion of qualitative yes-or-no belief. Fortunately

[2]I have to add a qualification due to Colin Howson (personal correspondence). Smullyan (1968) defines consistency for distributions of truth values. Consistency of sentences is then a derivative notion.

[3]If possibility theory (Zadeh 1978, Dubois & Prade 1988) is interpreted in terms of uncertainty rather than imprecision, one can define a notion of belief (positive degree of necessity) that is consistent and deductively closed in the finite, though not in the countable sense.

there is: ranking theory (Spohn 1988; 1990; to appear; manuscript). A function ϱ from a field of propositions $\mathcal{A}$ over a set of possibilities W into the set of natural numbers N enriched by ∞, $\varrho : \mathcal{A} \rightarrow N \cup \{\infty\}$, is a (*finitely minimitive*) *ranking function on* $\mathcal{A}$ iff for all $A, B \in \mathcal{A}$,

1. $\varrho(\emptyset) = \infty, \quad \varrho(W) = 0,$
2. $\varrho(A \cup B) = \min\{\varrho(A), \varrho(B)\}.$

A ranking function ϱ on a σ-field / complete field $\mathcal{A}$ is *countably* / *completely minimitive* iff for all countable / uncountable $\mathcal{B} \subseteq \mathcal{A}$,

$$\varrho(\bigcup \mathcal{B}) = \min\{\varrho(B) : B \in \mathcal{B}\}.$$

For $A \in \mathcal{A}$, the *conditional ranking function* $\varrho(\cdot \mid A) : \mathcal{A} \setminus \{\emptyset\} \rightarrow N \cup \{\infty\}$ based on the ranking function $\varrho(\cdot) : \mathcal{A} \rightarrow N \cup \{\infty\}$ is defined as

$$\varrho(\cdot \mid A) = \begin{cases} \varrho(\cdot \cap A) - \varrho(A), & \text{if} \quad \varrho(A) < \infty, \\ 0, & \text{if} \quad \varrho(A) = \infty. \end{cases}$$

Further stipulating $\varrho(\emptyset \mid A) = \infty$ ensures that every conditional ranking function is a ranking function. A ranking function ϱ is *regular* iff $\varrho(A) < \varrho(\emptyset)$ for all non-empty $A \in \mathcal{A}$. As an aside, $\mathcal{A} \subseteq \wp(\mathcal{W})$ is a (finitary) field over W iff for all $A, B \in \mathcal{A}$: $W \in \mathcal{A}$, $\overline{A} \in \mathcal{A}$, and $A \cup B \in \mathcal{A}$. A field $\mathcal{A}$ is a σ- / complete field iff for all countable / uncountable $\mathcal{B} \subseteq \mathcal{A}$: $\bigcup \mathcal{B} \in \mathcal{A}$.

A function ϱ from a language $\mathcal{L}$, i.e. a set of well formed formulas containing $\top$ and being closed under negation and disjunction, into $N \cup \{\infty\}$ is a *ranking on* $\mathcal{L}$ iff for all $\alpha, \beta \in \mathcal{L}$,

0. $\models \alpha \leftrightarrow \beta \quad \Rightarrow \quad \varrho(\alpha) = \varrho(\beta),$
1. $\varrho(\neg\top) = \infty, \quad \varrho(\top) = 0,$
2. $\varrho(\alpha \vee \beta) = \min\{\varrho(\alpha), \varrho(\beta)\}.$

For $\alpha \in \mathcal{L}$, the *conditional ranking* $\varrho(\cdot \mid \alpha) : \mathcal{L} \setminus \{\beta \in \mathcal{L} :\models \neg\beta\} \rightarrow N \cup \{\infty\}$ based on the ranking $\varrho(\cdot) : \mathcal{L} \rightarrow N \cup \{\infty\}$ is defined as

$$\varrho(\cdot \mid \alpha) = \begin{cases} \varrho(\cdot \wedge \alpha) - \varrho(\alpha), & \text{if} \quad \varrho(\alpha) < \infty, \\ 0, & \text{if} \quad \varrho(\alpha) = \infty. \end{cases}$$

Again, stipulating $\varrho(\beta \mid \alpha) = \infty$ if $\models \neg\beta$ guarantees that $\varrho(\cdot \mid \alpha)$ is a ranking on $\mathcal{L}$ for every $\alpha \in \mathcal{L}$. ϱ is *regular* iff $\varrho(\alpha) < \varrho(\neg\top)$ for all consistent $\alpha \in \mathcal{L}$.

Spohn's original formulation is in terms of what I have elsewhere called pointwise ranking functions. A function $\kappa : W \rightarrow N \cup \{\infty\}$ is a *pointwise* *ranking function on* W iff $\kappa(\omega) = 0$ for at least one $\omega \in W$. Every pointwise ranking function κ on W induces a completely minimitive ranking function ϱ on any field $\mathcal{A}$ over W by defining for all $A \in \mathcal{A}$,

$$\varrho(\emptyset) = \infty, \quad \varrho(A) = \min\{\kappa(\omega) : \omega \in A\}.$$

The converse is not true. For the relation between ranking functions on a field of propositions and pointwise ranking functions on a set of possibilities one level below see Huber (2006).

A probability measure $\Pr : \mathcal{A} \to \{0\} \cup (0,1]$ can be epistemically interpreted as an agent's degree of belief function. It has the following properties. For all $A, B, A_i, B_i \in \mathcal{A}$, $i \in N$, with $\{A_i\}$ a partition of W,

- $\Pr(W) = 1$ and $\Pr(\emptyset) = 0$
- $\Pr(A \cup B) = \Pr(A) + \Pr(B)$, if $A \cap B = \emptyset$
- $A \subseteq B \quad \Rightarrow \quad \Pr(A) \leq \Pr(B)$
- $\Pr(A) + \Pr(\overline{A}) = 1, \quad \sum_i \Pr(A_i) = \Pr(\bigcup\{A_i\}) = 1$
- $\Pr(B \mid A) = \Pr(B \cap A) \div \Pr(A)$ if $\Pr(A) > 0$
- $\Pr(B) = \sum_i \Pr(B \mid A_i) \cdot \Pr(A_i)$
- $\Pr(\bigcap\{B_i\}) = \prod_i \Pr\left(B_i \mid \bigcap_{j<i}\{B_j\}\right), \quad \bigcap_{j<1}\{B_j\} = W$

The last clauses are unrestrictedly defined if we stipulate $\Pr(B \mid A) = 1$ for $\Pr(A) = 0$. Countable additivity is assumed for the countably infinite versions.

Let us replace 1 by 0, 0 by ∞, $>$ by $<$, $+$ and $\sum$ by min, $\cdot$ and $\prod$ by $+$ and $\sum$, and $\div$ by $-$, as well as neglect the exclusiveness condition for summations. Then we get the corresponding properties of a ranking function $\varrho : \mathcal{A} \to N \cup \{\infty\}$, which can be epistemically interpreted as an agent's degree of disbelief function. For all $A, B, A_i, B_i \in \mathcal{A}$, $i \in N$, with $\{A_i\}$ a partition of W,

- $\varrho(W) = 0$ and $\varrho(\emptyset) = \infty$
- $\varrho(A \cup B) = \min\{\varrho(A), \varrho(B)\}$, whether or not $A \cap B = \emptyset$
- $A \subseteq B \quad \Rightarrow \quad \varrho(A) \geq \varrho(B)$
- $\varrho(A) = 0$ or $\varrho(\overline{A}) = 0, \quad \min\{\varrho(A_i) : i \in N\} = \varrho(\bigcup\{A_i\}) = 0$
- $\varrho(B \mid A) = \varrho(B \cap A) - \varrho(A)$, if $\varrho(A) < \infty$
- $\varrho(B) = \min\{\varrho(A_i) + \varrho(B \mid A_i) : i \in N\}$
- $\varrho(\bigcap\{B_i\}) = \sum_i \varrho\left(B_i \mid \bigcap_{j<i}\{B_j\}\right), \quad \bigcap_{j<1}\{B_j\} = W$

Countable minimitivity is assumed for the countably infinite versions. It is useful to keep this translational device in mind, even though it is no more than that.

A ranking function ϱ over $\mathcal{A}$ is interpreted as an agent's degree of disbelief function for the propositions in $\mathcal{A}$. $\varrho(A)$ is the agent's degree of disbelief for A. It tells us how reluctant she is to give up her qualitative disbelief in A. $\varrho(\overline{A})$ is the agent's degree of disbelief for $\overline{A}$. It tells us how reluctant she is to give up her qualitative disbelief in $\overline{A}$. This will become important below.

The *belief function* $\beta(\cdot) : \mathcal{A} \to \{-\infty\} \cup Z \cup \{\infty\}$ associated with the ranking function $\varrho(\cdot) : \mathcal{A} \to N \cup \{\infty\}$ is defined as $\beta(\cdot) = \varrho(\bar{\cdot}) - \varrho(\cdot)$. The corresponding function in probability theory is $Be(\cdot) = \Pr(\cdot) \div \Pr(\bar{\cdot})$. Be is, for instance, used in Bayesian confirmation theory, where Milne (1996) argues that $r = \log[\Pr(H \mid E) \div \Pr(H)]$ is the one true measure of confirmation, whereas Fitelson (1999) says it is $l = \log[Be(H \mid E) \div Be(H)]$.

A ranking function ϱ on a field $\mathcal{A}$ induces a set of propositions $Bel_\varrho \subseteq \mathcal{A}$, ϱ's *belief set*:

$$Bel_\varrho = \left\{A \in \mathcal{A} : \varrho\left(\overline{A}\right) > \varrho(A)\right\} = \left\{A \in \mathcal{A} : \varrho\left(\overline{A}\right) > 0\right\}.$$

The belief set Bel_ϱ is the set of all propositions or sentences whose complements or negations the agent disbelieves to some positive degree. Alternatively we can say that an agent's belief set is the set of all propositions she believes to some positive degree, since $Bel_\varrho = \{A \in \mathcal{A} : \beta(A) > 0\}$, where β is the belief function associated with ϱ. Note that one and the same belief set may be induced by many different ranking functions.

3. The Consistency Argument

The second question is whether something along the following lines is true. A degree of disbelief function satisfies the ranking axioms iff the corresponding belief set is consistent and deductively closed. The theorem in the next section states something along these lines. Let us first get clear about the structure of the argument for the thesis that an agent's degrees of disbelief should obey the ranking calculus.

A (depragmatized) Dutch Book Argument is supposed to vindicate probabilism, the thesis that an agent's degrees of belief should obey the probability calculus. It has the following ingredients.

0. (Fair) betting ratios.
1. A link between degrees of belief and (fair) betting ratios. Sometimes degrees of belief are *defined* as (fair) betting ratios. Then this link is

identity. Sometimes degrees of belief are *measured* by (fair) betting ratios. Then this link is weaker than identity. In the latter case we are facing the *connection problem*, the question of how degrees of belief and (fair) betting ratios are related to each other.

2. The (depragmatized) *Dutch Book Principle*. It says that it is pragmatically (epistemically) defective to accept a series of bets which guarantees a sure loss, i.e. a *Dutch Book* (to consider a Dutch Book to be fair).

3. The (depragmatized) *Dutch Book Theorem*. It says that an agent's (fair) betting ratios obey the probability calculus iff the agent never accepts a Dutch Book (never considers a Dutch Book to be fair).

4. A conclusion: it is pragmatically (epistemically) defective to have degrees of belief that violate the probability axioms.

A (depragmatized) Dutch Book Argument is an argument with premises 1-3 and conclusion 4. Particular arguments differ from each other by the exact form the above ingredients take. Obviously, if we strengthen the link between degrees of belief and (fair) betting ratios we have less of a problem in getting from 2 and 3 to 4, but we have more problems in making 1 plausible.

There are many objections to the vindication of probabilism by a Dutch Book Argument (Hájek 2005; to appear). For instance, Dutch-Book-*ability* is a mere possibility, often far from the agent being actually Dutch-Book-*ed*. The corresponding feature is shared by the Consistency Argument. However, the purpose of this is not to rigorously discuss Dutch Book Arguments. The aim is to name the components of the argument, and the role they play therein, in order to motivate the corresponding argument for the *ranking thesis* that an agent's degrees of disbelief should obey the ranking axioms. I do, however, want to draw the reader's attention to one point: the distinction between pragmatic and epistemic defectiveness, or practical and theoretical rationality, most famously discussed by Kant (1902), but also present in other areas of formal epistemology (Rott 2001).

What are the corresponding ingredients in the *Consistency Argument*? Clearly, ranks play the role of probabilities, and the *Consistency Theorem* of the next section will be the substitute for the Dutch Book Theorem. The conclusion will be that it is epistemically defective to have degrees of disbelief that violate the ranking axioms. The two ingredients differing in important ways are the substitutes for (fair) betting ratios and the Dutch Book Principle. These are, respectively, degrees of entrenchment and a principle of *theoretical* rationality saying that it is *epistemically* defective to have beliefs that are not both consistent and deductively closed.

The idea that ranks can be measured by the agent's contraction behavior is developed in Spohn (1999), although in Spohn (to appear) Matthias Hild is said to have presented it first and independently. Spohn (manuscript) presents three methods for the measurement of ranks. I will only use what he calls the method of enhancements. To give the reader an idea of how this measurement works, suppose I disbelieve that Sacramento is the capital of California, $\overline{S} \in Bel_{Franz}$. Then my rank for S, $\varrho_{Franz}(S)$, can be measured as follows. We put me on a busy street, say Sunset Boulevard on a Saturday night, and count the number of people who pass by and tell me that Sacramento is the capital of California. My rank for S equals n precisely if I stop disbelieving S after exactly n people have passed by and told me S. So $\varrho_{Franz}(S)$ is measured by the number of "independent and minimally positively reliable information sources" saying S that it takes for me to give up my disbelief in S. If I do not disbelieve S to begin with, my rank for S is 0.

The relation between degrees of disbelief and degrees of entrenchment is a delicate one, much like the relation between degrees of belief and (fair) betting ratios. One option is to take the former as primitive (Eriksson & Hájek to appear), and to say that the latter measure them under suitable conditions. Another option is to "go hypothetical": my degrees of disbelief are the degrees of entrenchment that I would have if there were an infinite stock of independent and minimally positively reliable information sources at my disposal. I think the second option is attractive, and more attractive than its probabilistic counterpart. It is most attractive if we base it on a ranktheoretic theory of counterfactuals. The prospects for such an account are good: a ranking function for each world replaces the selection functions of Stalnaker (1968) or the similarity ordering of Lewis (1973). The ranking function of a world is objectively interpreted as the ranktheoretic analogue of chance in a world in probability theory or similarity with respect to a world in the theory of counterfactuals.

Here is the plan for the rest of the paper. The next section states the *Consistency Theorem* which says that an agent's degrees of entrenchment satisfy the ranking axioms iff the agent's belief set is and will always be consistent and deductively closed, provided updating leads from one ranking function to another. Section 5 reformulates this in terms of conditional consistency. Section 6 presents various probabilistic and ranktheoretic update rules. Section 7 contains the consistency theorems for these, thus vindicating the proviso that updating leads from one ranking function to another. I will conclude in section 8.

4. The Consistency Theorem

An agent's degree of entrenchment for a proposition A is defined as the number of independent and minimally positively (mp), and hence equally, reliable information sources saying A that it takes for the agent to give up her disbelief that A. If the agent does not disbelieve A to begin with, it does not take any information source saying A to make her stop disbelieving A. So her degree of entrenchment for A is 0. If no finite number of information sources is able to make an agent stop disbelieving A, her degree of entrenchment for A is ∞. To receive the information A is, among others, to also receive the information B, for any proposition $B \supseteq A$. To independently and mp-reliably receive n times the information A is, among others, to independently and mp-reliably receive n times the information $B \supseteq A$. It is *not* to independently and mp-reliably receive m times the information B, for some $m \neq n$. The reason is that the number n characterizes the reliability of the information source saying A. That source is the same for any logical consequence of A. If you tell me that the temperature today at noon will be 93° Fahrenheit, you also tell me that the temperature today at noon will be between 90° and 96° Fahrenheit. But it is still *you* who tells me so. Therefore the reliability with which I get the second information is *exactly* the same as the reliability with which I get the first information. The difference between the two is a difference in content. This will become important in the proof of the Consistency Theorem (3.2 below).

When we measure ranks we count information sources. For the measurement to work, these have to be independent and mp-reliable. Of course, one person's saying A will sometimes make somebody stop disbelieving A, while the sermons of twenty others won't. And my father's telling me A after my mother has already explained to me why A won't make much of difference for me either. What's new? Neither are my (fair) betting ratios always independent of the truth values of the propositions I am betting on, nor are they never affected by the stakes at issue. The operational surrogate is not the theoretical entity itself. Often it does not even provide a good measurement.

The agent's belief set at a given time is the set of propositions whose complements she disbelieves according to her entrenchment function. Combined with an update rule, this entrenchment function specifies what the agent will believe if she receives new information. Each new item of information gives thus rise to a new entrenchment function and, accordingly, a new belief set.

DEFINITION 4.1. Let $\mathcal{L}$ be a language, let $\mathcal{A}$ be a field over a set of possibilities W, let $\Gamma \subseteq \mathcal{L}$ be a set of sentences, and let $\mathcal{B} \subseteq \mathcal{A}$ be a set of propositions.

Γ is *consistent* iff each finite $\Gamma_{fin} \subseteq \Gamma$ has a model. Γ is *deductively closed* iff for each finite $\Gamma_{fin} \subseteq \Gamma$ and all $\alpha \in \mathcal{L}$: if $\Gamma_{fin} \models \alpha$, then $\alpha \in \Gamma$.

$\mathcal{B}$ is *consistent in the finite / countable / complete sense* iff for each finite / countable / uncountable $\mathcal{B}' \subseteq \mathcal{B}$, $\bigcap \mathcal{B}' \neq \emptyset$. $\mathcal{B}$ is *deductively closed in the finite / countable / complete sense* iff for each finite / countable / uncountable $\mathcal{B}' \subseteq \mathcal{B}$ and all $A \in \mathcal{A}$: if $\bigcap \mathcal{B}' \subseteq A$, then $A \in \mathcal{B}$.

THEOREM 4.2 (Consistency Theorem). *Let $\mathcal{L}$ be a language, and let $\mathcal{A}$ be a field of propositions over the set of possibilities W.*

An agent's entrenchment function ϱ on $\mathcal{L}$ is a ranking on $\mathcal{L}$ iff every possible current or future belief set $Bel \subseteq \mathcal{L}$ based on ϱ is consistent and deductively closed. Updating is temporarily assumed to lead from one ranking on $\mathcal{L}$ to another ranking on $\mathcal{L}$.

An agent's entrenchment function ϱ on $\mathcal{A}$ is a finitely / countably / completely minimitive ranking function on $\mathcal{A}$ iff every possible current or future belief set $Bel \subseteq \mathcal{A}$ based on ϱ is consistent and deductively closed in the finite / countable / complete sense. Updating is temporarily assumed to lead from one finitely / countably / completely minimitive ranking function on $\mathcal{A}$ to another finitely / countably / completely minimitive ranking function on $\mathcal{A}$.

Degrees of entrenchment are assumed to be numbers from $N \cup \{\infty\}$.

PROOF. $\Rightarrow$: Let $\mathcal{A}$ be a field of propositions over the set of possibilities W. Suppose an agent's entrenchment function ϱ on $\mathcal{A}$ is a finitely / countably / completely minimitive ranking function.
By definition, $Bel_\varrho = \left\{A \in \mathcal{A} : \varrho\left(\overline{A}\right) > 0\right\}$. Finite / countable / complete minimitivity yields for each finite / countable / uncountable $\mathcal{B} \subseteq Bel_\varrho$,

$$\varrho\left(\bigcup \mathcal{B}^{neg}\right) = \min\left\{\varrho\left(\overline{A}\right) : A \in Bel_\varrho\right\} > 0, \quad \mathcal{B}^{neg} = \left\{\overline{A} \in \mathcal{A} : A \in \mathcal{B}\right\}.$$

Hence $\varrho\left(\overline{\bigcup \mathcal{B}^{neg}}\right) = \varrho\left(\bigcap \mathcal{B}\right) = 0$, and so $\bigcap \mathcal{B} \neq \emptyset$. Thus Bel_ϱ is consistent in the finite / countable / complete sense.

Furthermore, let $\mathcal{B} \subseteq Bel_\varrho$ be finite / countable / uncountable, and suppose $\bigcap \mathcal{B} \subseteq A$ for some $A \in \mathcal{A}$. We have to show that $A \in Bel_\varrho$, i.e. $\varrho\left(\overline{A}\right) > 0$. By finite / countable / complete minimitivity, $\min\left\{\varrho\left(\overline{B}\right) : B \in \mathcal{B}\right\} = \varrho\left(\bigcup \mathcal{B}^{neg}\right) > 0$. So $\varrho\left(\overline{A}\right) > 0$, because $\overline{A} \subseteq \bigcup \mathcal{B}^{neg}$ and ϱ is monotonic w.r.t. set inclusion. Hence Bel_ϱ is deductively closed in the finite / countable / complete sense. Similarly for a ranking ϱ on a language $\mathcal{L}$. Note, though, that compactness, which has been built into the definitions of consistency and deductive closure, is needed. Otherwise we face the problem that $\bigwedge \Gamma$ and $\bigvee \Gamma$ are not defined for infinite $\Gamma \subseteq \mathcal{L}$.

As this holds true for any ranking function, the belief set Bel_{ϱ^+} of any possible future entrenchment function ϱ^+ is also consistent and deductively closed in the finite / countable / complete sense, provided updating leads from one finitely / countably / completely minimitive ranking function to another finitely / countably / completely minimitive ranking function. Gärdenfors & Rott (1995: 37) call this the principle of categorical matching[4], and (subject to cosmetic adjustments in the case of Shenoy revision) it is true for all ranktheoretic update rules considered in section 6.

$\Leftarrow$: Let $\mathcal{A}$ be a field of propositions over the set of possibilities W, and let ϱ be an agent's entrenchment function on $\mathcal{A}$.

(1) Suppose $\varrho(W) > 0$. Then $\emptyset \in \{A \in \mathcal{A} : \varrho(\overline{A}) > 0\} = Bel_\varrho$, and so $\bigcap \mathcal{B} = \emptyset$ for at least one finite $\mathcal{B} \subseteq Bel_\varrho$. Thus, Bel_ϱ is finitely, countably, and completely inconsistent. Similarly for a ranking ϱ on a language $\mathcal{L}$.

(2) Suppose next $\varrho(\emptyset) < \infty$. Assume the agent receives evidence equivalent to being told $\emptyset$ by n or more independent and mp-reliable information sources. By the definition of degrees of entrenchment, the resulting entrenchment function ϱ_n after independently and mp-reliably receiving n or more times the information $\emptyset$ is such that $\varrho_n(\emptyset) = 0$. Therefore $W \notin \{A \in \mathcal{A} : \varrho_n(\overline{A}) > 0\} = Bel_{\varrho_n}$. As $\bigcap \mathcal{B} \subseteq W$ for each $\mathcal{B} \subseteq Bel_{\varrho_n}$, Bel_{ϱ_n} is not deductively closed in the finite, countable, or complete sense. Similarly for a ranking ϱ on a language $\mathcal{L}$.

(3) Now suppose $\mathcal{A}$ is a finitary / σ- / complete field over W, and ϱ on $\mathcal{A}$ violates finite / countable / complete minimitivity.

(3.1) Suppose first there is a finite / countable / uncountable $\mathcal{B} \subseteq \mathcal{A}$ such that $\varrho(\bigcup \mathcal{B}) < \min\{\varrho(A) : A \in \mathcal{B}\}$. If $\varrho(\bigcup \mathcal{B}) = 0$, we have

$$\forall A \in \mathcal{B} : \overline{A} \in Bel_\varrho, \quad \bigcap \mathcal{B}^{neg} = \overline{\bigcup \mathcal{B}} \notin Bel_\varrho, \quad \mathcal{B}^{neg} = \{\overline{A} \in \mathcal{A} : A \in \mathcal{B}\},$$

which means that Bel_ϱ is not deductively closed in the finite / countable / complete sense. If $\varrho(\bigcup \mathcal{B}) = n > 0$, we have the following situation:

|————-|——————————|———————————

0 $\quad \varrho(\bigcup \mathcal{B}) = n \quad \min\{\varrho(A) : A \in \mathcal{B}\} = n' > n$

Assume the agent receives evidence equivalent to being told $\bigcup \mathcal{B}$ by at least n, but fewer than n' independent and mp-reliable information sources. The resulting entrenchment function ϱ_{n^*} after independently and mp-reliably receiving $n^* \in [n, n')$ times the information $\bigcup \mathcal{B}$ is such that $\varrho_{n^*}(\bigcup \mathcal{B}) = 0$.

[4]Spohn (1988: sct. 3) argues that AGM belief revision theory is incapable of iterated revisions precisely because it violates this principle. In order to revise a belief set one needs an entrenchment ordering. The result of a first AGM revision does not give rise to a new entrenchment ordering, but merely to a new belief set. So there is no second AGM revision.

Now even if for all $A \in Bel_\varrho$, $\varrho_{n^*}(A) = \varrho(A) - n^* > 0$, we have the following situation:

|——————————————|——————————————————

$\varrho_{n^*}(\bigcup \mathcal{B}) \leq 0$ $\min\{\varrho_{n^*}(A) : A \in \mathcal{B}\} > 0$

Hence

$$\forall A \in \mathcal{B} : \overline{A} \in Bel_{\varrho_{n^*}}, \quad \bigcap \mathcal{B}^{neg} = \overline{\bigcup \mathcal{B}} \notin Bel_{\varrho_{n^*}},$$

which means that $Bel_{\varrho_{n^*}}$ is not deductively closed in the finite / countable / complete sense. Similarly for a ranking ϱ on a language $\mathcal{L}$.

(3.2) Now suppose there is a finite / countable / uncountable $\mathcal{B} \subseteq \mathcal{A}$ such that $\varrho(\bigcup \mathcal{B}) > \min\{\varrho(A) : A \in \mathcal{B}\}$. If $\min\{\varrho(A) : A \in \mathcal{B}\} = \varrho(A^*) = 0$, we have

$$\overline{A^*} \notin Bel_\varrho, \quad \bigcap \mathcal{B}^{neg} = \overline{\bigcup \mathcal{B}} \in Bel_\varrho,$$

which means that Bel_ϱ is not deductively closed in the finite / countable / complete sense. If $\min\{\varrho(A) : A \in \mathcal{B}\} = \varrho(A^*) = n > 0$, for some $A^* \in \mathcal{B}$, we have the following situation:

|—————-|—————————————————-|——————————-

0 $\varrho(A^*) = \min\{\varrho(A) : A \in \mathcal{B}\} = n$ $\varrho(\bigcup \mathcal{B}) = n' > n$

Assume the agent receives evidence equivalent to being told A^* by at least n, but fewer than n' independent and mp-reliable information sources. The resulting entrenchment function ϱ_{n^*} after independently and mp-reliably receiving $n^* \in [n, n')$ times the information A^* is such that $\varrho_{n^*}(A^*) = 0$. Now $\varrho_{n^*}(\bigcup \mathcal{B}) \geq \varrho(\bigcup \mathcal{B}) - n^* > 0$, and so we have the following situation:

|————————|——————————-

$\varrho_{n^*}(A^*) \leq 0$ $\varrho_{n^*}(\bigcup \mathcal{B}) > 0$

Hence

$$\overline{A^*} \notin Bel_{\varrho_n}, \quad \bigcap \mathcal{B}^{neg} = \overline{\bigcup \mathcal{B}} \in Bel_{\varrho_n},$$

which means that Bel_{ϱ_n} is not deductively closed in the finite / countable / complete sense. Similarly for a ranking ϱ on a language $\mathcal{L}$. ■

As it is obvious how to extend the following to rankings ϱ on languages $\mathcal{L}$, these are not considered anymore.

5. The Consistency Theorem Reformulated

Observe with Spohn (to appear)

OBSERVATION 5.1 (Spohn's Observation). *Let $\mathcal{A}$ be a field over the set of possibilities W, and let ϱ be a finitely / countably / completely minimitive ranking function on $\mathcal{A}$ with $\varrho(\cdot \mid \cdot)$ as its conditional ranking function.*

Then we have for all finite / countable / uncountable $\mathcal{B} \subseteq \mathcal{A}$ *with* $\bigcup \mathcal{B} \neq \emptyset$,

$$\min \left\{ \varrho \left(A \mid \bigcup \mathcal{B} \right) : A \in \mathcal{B} \right\} = 0,$$

and for all $A, B \in \mathcal{A}$ *with* $B \neq \emptyset$,

$$\varrho (A \mid W) = \varrho (A), \quad \varrho (\emptyset \mid A) = \infty, \quad \varrho (B \mid A) = \varrho (B \cap A) - \varrho (A).$$

Conversely, let $\varrho (\cdot \mid \cdot) : \mathcal{A} \times \mathcal{A} \to N \cup \{\infty\}$ *and* ϱ *be functions with these properties. Then* ϱ *is a finitely / countably / completely minimitive ranking function on* $\mathcal{A}$ *with* $\varrho (\cdot \mid \cdot)$ *as its conditional ranking function, i.e. for all finite / countable / uncountable* $\mathcal{B} \subseteq \mathcal{A}$,

$$\varrho (W) = 0, \quad \varrho (\emptyset) = \infty, \qquad \varrho \left(\bigcup \mathcal{B} \right) = \min \{ \varrho (A) : A \in \mathcal{B} \}$$

and for all $A, B \in \mathcal{A}$ *with* $B \neq \emptyset$,

$$\varrho (B \mid A) = \varrho (B \cap A) - \varrho (A), \quad \varrho (\emptyset \mid A) = \infty.$$

PROOF. $\Leftarrow$: The first equation is called conditional consistency. We get it from minimitivity and the definition of conditional ranking functions. The assumption that $\bigcup \mathcal{B} \neq \emptyset$ enters in the last step.

$$\begin{aligned} \varrho \left(\bigcup \mathcal{B} \right) = \min \{ \varrho (A) : A \in \mathcal{B} \} \quad & \Leftrightarrow \quad \min \{ \varrho (A) : A \in \mathcal{B} \} - \varrho \left(\bigcup \mathcal{B} \right) = 0 \\ & \Leftrightarrow \quad \min \left\{ \varrho (A) - \varrho \left(\bigcup \mathcal{B} \right) : A \in \mathcal{B} \right\} = 0 \\ & \Leftrightarrow \quad \min \left\{ \varrho \left(A \mid \bigcup \mathcal{B} \right) : A \in \mathcal{B} \right\} = 0 \end{aligned}$$

By the definition of a ranking function, $\varrho (W) = 0$ and $\varrho (\emptyset) = \infty$. By the definition of a conditional ranking function we have for non-empty $B \in \mathcal{A}$, $\varrho (B \mid W) = \varrho (B \cap W) - \varrho (W)$ and $\varrho (\emptyset \mid W) = \infty$. Combining these equations yields $\varrho (A \mid W) = \varrho (A)$ for all $A \in \mathcal{A}$. The last two equations to be established are the defining clauses of conditional ranking functions.
$\Rightarrow$: For non-empty $\bigcup \mathcal{B}$ we have already seen the equivalence of conditional consistency and the minimitivity axiom. $\varrho (\emptyset) = \infty$ follows from $\varrho (A \mid W) = \varrho (A)$ and $\varrho (\emptyset \mid A) = \infty$. It entails the minimitivity axiom for empty $\bigcup \mathcal{B}$. As $\varrho (A \mid W) = \varrho (A)$, the range of ϱ is a subset of the range of $\varrho (\cdot \mid \cdot)$. Together $\varrho (A \mid W) = \varrho (A)$ and $\varrho (B \mid A) = \varrho (B \cap A) - \varrho (A)$ entail $\varrho (W) = \varrho (W) - \varrho (W)$. The last equation is equal to 0, because we have assumed that $\infty + -\infty = 0$. The final two clauses to be established hold by assumption. ■

As every ranking function, a conditional ranking function $\varrho(\cdot \mid E)$ induces a belief set $Bel_{\varrho(\cdot|E)}$. $Bel_{\varrho(\cdot|E)}$ is called a conditional belief set of ϱ, viz. ϱ's belief set conditional on E. So ϱ's belief set is its belief set conditional on W, $Bel_\varrho = Bel_{\varrho(\cdot|W)}$. $\{W\}$ is ϱ's belief set conditional on any $E \in \mathcal{A}$ with $\varrho(E) = \infty$, including the empty set $\emptyset$. Thus, rather than believing everything conditional on a proposition that she thinks is impossible, an epistemic agent refrains from believing anything except the tautology in such a case. If one holds that even this is believing too much given a (supposedly) contradictory condition, one can resort to one of the following two options. Either one sticks to $\varrho(A \mid \emptyset) = 0$ for all A *including* $\emptyset$ ($\varrho(A \mid B) = 0$ for all A including $\emptyset$, and B with $\varrho(B) = \infty$) and takes the empty set as the resulting belief set; or one restricts $\varrho(\cdot \mid E)$ to non-empty E (or to E with a finite rank). Note, though, that the empty set is not deductively closed; and the function assigning 0 to all propositions including the empty set is not a ranking function.

Theorem 5.2 (Conditional Consistency Theorem). *Let $\mathcal{A}$ be a field of propositions over the set of possibilities W. An agent's entrenchment function ϱ on $\mathcal{A}$ is a finitely / countably / completely minimitive ranking function iff all of ϱ's conditional belief sets are consistent and deductively closed in the finite / countable / complete sense.*

Conditional degrees of entrenchment are assumed to be numbers from $N \cup \{\infty\}$, and to be (defined as) differences of unconditional degrees of entrenchment. Unconditional degrees of entrenchment are assumed to be (defined as) degrees of entrenchment conditional on W. The conditional degree of entrenchment for the empty set is assumed to be (defined as) ∞, for any condition including the empty set. That is, $\varrho(\cdot \mid \cdot)$ is assumed to be an $N \cup \{\infty\}$-valued function on $\mathcal{A} \times \mathcal{A}$ such that for $A, B \in \mathcal{A}$ with $B \neq \emptyset$,

$$\varrho(A \mid W) = \varrho(A), \quad \varrho(\emptyset \mid A) = \infty, \quad \varrho(B \mid A) = \varrho(B \cap A) - \varrho(A).$$

Proof. $\Rightarrow$: Let $\mathcal{A}$ be a field of propositions over the set of possibilities W, and let $E \in \mathcal{A}$. Suppose an agent's entrenchment function ϱ on $\mathcal{A}$ is a finitely / countably / completely minimitive ranking function. If $\varrho(E) < \infty$, then $\varrho(\cdot \mid E)$ is a finitely / countably / completely minimitive ranking function. So $Bel_{\varrho(\cdot|E)}$ is consistent and deductively closed in the finite / countable / complete sense. If $\varrho(E) = \infty$, then $\varrho(B \mid E) = 0$ for $\emptyset \neq B \in \mathcal{A}$, and $\varrho(\emptyset \mid E) = \infty$. Hence $Bel_{\varrho(\cdot|E)} = \{W\}$, which is consistent and deductively closed in the finite, countable, and complete sense.
$\Leftarrow$: Let $\mathcal{A}$ be a field of propositions over the set of possibilities W, let ϱ be an agent's entrenchment function on $\mathcal{A}$, and let $\mathcal{B} \subseteq \mathcal{A}$ be finite

/ countable / uncountable. Suppose $\min\{\varrho(A \mid \bigcup\mathcal{B}) : A \in \mathcal{B}\} > 0$. Then

$$\forall A \in \mathcal{B} : \overline{A} \in Bel_{\varrho(\cdot|\bigcup\mathcal{B})}, \quad \bigcap \mathcal{B}^{neg} = \overline{\bigcup \mathcal{B}} \notin Bel_{\varrho(\cdot|\bigcup\mathcal{B})},$$

where $\mathcal{B}^{neg} = \left\{\overline{A} \in \mathcal{A} : A \in \mathcal{B}\right\}$. This means that $Bel_{\varrho(\cdot|\bigcup\mathcal{B})}$ is not deductively closed in the finite / countable / complete sense. ■

The reformulation of ranking functions turns the conditional consistency theorem into a triviality. However, a lot is hidden in the assumptions, which are not covered by the consistency vindication.

First, it is not clear how to vindicate $\varrho(A \mid W) = \varrho(A)$ for all $A \in \mathcal{A}$ by a consistency argument, because it concerns the relation of two different functions. There is relief insofar as this assumption can be replaced by the condition that $\varrho(W) = 0$. As we have seen in the proof of the first theorem, the latter condition can be vindicated by a consistency argument. But a formulation of ranking functions purely in terms of restrictions on the conditional functions, and a definitional link between conditional and unconditional ranking functions as above would be preferable.

Second, as in the definition of conditional ranking functions, we have to assume $\varrho(\emptyset \mid A) = \infty$.

The third assumption gives most content to the link between conditional and unconditional ranking functions: $\varrho(B \mid A) = \varrho(B \cap A) - \varrho(A)$. Together with conditional consistency it entails the monotonicity of unconditional ranking functions w.r.t. set inclusion,

$$A \subseteq B \quad \Rightarrow \quad \varrho(A) \geq \varrho(B).$$

This in turn is equivalent to half of the minimitivity condition, viz.

$$\varrho\left(\bigcup \mathcal{B}\right) \leq \min\{\varrho(A) : A \in \mathcal{B}\}.$$

As in the consistency theorem I have assumed that conditional, and hence unconditional, degrees of entrenchment take values from $N \cup \{\infty\}$. Together with the definitional link between conditional and unconditional entrenchment functions, this also entails the monotonicity w.r.t. set inclusion of ranking functions.

There is a final worry about the vindication of the ranking thesis in terms of conditional consistency. It does not carry over to update rules.

6. Conditionalization and Revision

If evidence comes in form of a proposition only, probabilism is extended by

UPDATE RULE 6.1 (Strict Conditionalization). If $\Pr(\cdot) : \mathcal{A} \to \Re$ is your subjective probability measure at time t and between t and t' you become certain of $E \in \mathcal{A}$ and no logically stronger proposition, then your subjective probability measure at time t' should be $\Pr_E(\cdot) : \mathcal{A} \to \Re$,

$$\Pr{}_E(\cdot) = \Pr(\cdot \mid E) = \Pr(\cdot \cap E) \div \Pr(E) \quad \text{if} \quad \Pr(E) > 0.$$

Note that strict conditionalization does not satisfy the principle of categorical matching. If $\Pr(E) = 0$, then $\Pr(\cdot \mid E)$ is not defined. Even for regular Pr, and even if E is assumed to be non-empty, strict conditionalization does not strictly satisfy the principle of categorical matching. $\Pr(\cdot \mid E)$ is not regular when Pr is, but only regular conditional on E, i.e. such that $\Pr(A \mid E) > 0$ for all $A \in \mathcal{A}$ with $A \cap E \neq \emptyset$ rather than for all $A \in \mathcal{A}$ with $A \neq \emptyset$.

The corresponding update rule in ranking theory is

UPDATE RULE 6.2 (Plain Conditionalization). If $\varrho(\cdot) : \mathcal{A} \to N \cup \{\infty\}$ is your ranking function at time t and between t and t' you become certain of $E \in \mathcal{A}$ and no logically stronger proposition, then your ranking function at time t' should be $\varrho_E(\cdot) : \mathcal{A} \to N \cup \{\infty\}$, $\varrho_E(\cdot) = \varrho(\cdot \mid E)$, where for all $B \in \mathcal{A}$ with $B \neq \emptyset$,

$$\varrho_E(B) = \varrho(B \cap E) - \varrho(E) \quad \text{and} \quad \varrho(\emptyset \mid E) = \infty,$$

and $\infty - n = \infty$ and $\infty + -\infty = 0$.

Plain conditionalization satisfies the principle of categorical matching and leads from one ranking function to another ranking function. As with strict conditionalization, regular ranking functions are only turned into conditionally regular ranking functions.

Jeffrey's insight is that evidence usually does not come in form of a proposition only. Strictly speaking, we (almost) never become certain of a proposition. Rather, evidence is such that it changes our degrees of belief for the propositions of a partition of the set of possibilities. In this case probabilism is extended by

UPDATE RULE 6.3 (Jeffrey Conditionalization). If $\Pr(\cdot) : \mathcal{A} \to \Re$ is your subjective probability measure at time t and between t and t' your subjective probabilities on the partition $\{E_i \in \mathcal{A} : i \in I\}$ change to $p_i \in [0, 1]$

with $\sum_i p_i = 1$ ($p_i = 0$ for $\Pr(E_i) = 0$ and $p_i = 1$ for $\Pr(E_i) = 1$), and your positive subjective probabilities change on no finer partition, then your subjective probability measure at time t' should be $\Pr_{E_i \to p_i}(\cdot) : \mathcal{A} \to \Re$,

$$\Pr{}_{E_i \to p_i}(\cdot) = \sum{}_i \Pr(\cdot \mid E_i) \cdot p_i.$$

See Jeffrey (1983).

As dictated by the translational device from section 2, the ranktheoretic analogue is

UPDATE RULE 6.4 (Spohn Conditionalization). If $\varrho(\cdot) : \mathcal{A} \to N \cup \{\infty\}$ is your ranking function at time t and between t and t' your ranks on the partition $\{E_i \in \mathcal{A} : i \in I\}$ change to $n_i \in N \cup \{\infty\}$ with $\min_i \{n_i\} = 0$ ($n_i = \infty$ for $E_i = \emptyset$ and $n_i = 0$ for $E_i = W$), and your finite ranks change on no finer partition, then your ranking function at time t' should be $\varrho_{E_i \to n_i}(\cdot) : \mathcal{A} \to N \cup \{\infty\}$,

$$\varrho_{E_i \to n_i}(\cdot) = \min{}_i \{\varrho(\cdot \mid E_i) + n_i\}.$$

As the reader will have noticed, Spohn conditionalization is more general than Jeffrey conditionalization in two respects. First, the parameters n_i are required to be 0 and ∞ only for W and $\emptyset$, respectively, whereas the parameters p_i are required to be 1 and 0 for all E_i (not only W and $\emptyset$) with $\Pr(E_i) = 1$ and $\Pr(E_i) = 0$, respectively. Second, in the probabilistic case the indices i range de facto over the natural numbers N, because there can only be countable many positive parameters p_i (otherwise $\sum_i p_i > 1$). In the ranktheoretic case the condition $\min_i \{n_i\} = 0$ does not impose such a restriction.

It is important to note that the parameters p_i and n_i characterize the agent's posterior degree of belief and disbelief in the E_i, respectively. They do not characterize the evidential impact on the agent's epistemic state of what happens between t and t'. Jeffrey and Spohn conditionalization focus on the result of the update process. Therefore $\Pr_{E_i \to p_i}(E_i) = p_i$ and $\varrho_{E_i \to n_i}(E_i) = n_i$. In this sense they do not characterize the evidence as such. Rather, they characterize the result at time t' of the interaction between the prior degree of (dis)belief function at time t and the evidence received between t and t'.

Jeffrey's suggestion is that evidence might even come in form of a new degree of belief function over a subfield of the original field of propositions. In this case probabilism is extended by

UPDATE RULE 6.5 (Jeffrey Revision). If $\Pr(\cdot) : \mathcal{A} \to \Re$ is your subjective probability measure at time t and between t and t' your subjective probability measure on the field $\mathcal{E} \subseteq \mathcal{A}$ changes to $\mathrm{Pr}'(\cdot) : \mathcal{E} \to \Re$ ($\mathrm{Pr}'(E) = \mathrm{Pr}(E)$ if $\mathrm{Pr}(E) \in \{0, 1\}$), and the positive part of your subjective probability measure changes on no field $\mathcal{B}$ with $\mathcal{E} \subset \mathcal{B} \subseteq \mathcal{A}$, then your subjective probability measure at time t' should be $\mathrm{Pr}_{\mathrm{Pr} \to \mathrm{Pr}'}(\cdot) : \mathcal{A} \to \Re$,

$$\mathrm{Pr}_{\mathrm{Pr} \to \mathrm{Pr}'}(\cdot) = \sum\nolimits_i \mathrm{Pr}(\cdot \mid E_i) \cdot \mathrm{Pr}'(E_i),$$

where $\{E_i \in \mathcal{E} : i \in N\}$ is a set of exclusive propositions with $\mathrm{Pr}'(E_i) > 0$ for all $i \in N$ for which there is no superset $\{B_j \in \mathcal{E} : j \in N\}$ of exclusive propositions such that $\mathrm{Pr}'(B_j) > 0$ for all $j \in N$.

Jeffrey revision satisfies the principle of categorical matching, because it is assumed that propositions with extreme probabilities keep their extreme probabilities. If we start with a regular probability measure, Jeffrey conditionalization leads to another regular probability measure, provided we update by a regular probability measure. Strict coherence (Shimony 1955) urges us to do so whenever we can (but we can't always).

The ranktheoretic analogue is

Update Rule 6.6 (Spohn Revision). If $\varrho(\cdot) : \mathcal{A} \to N \cup \{\infty\}$ is your ranking function at time t and between t and t' your ranking function on the field $\mathcal{E} \subseteq \mathcal{A}$ changes to $\varrho'(\cdot) : \mathcal{E} \to N \cup \{\infty\}$, and the finite part of your ranking function changes on no field $\mathcal{B}$ with $\mathcal{E} \subset \mathcal{B} \subseteq \mathcal{A}$, then your ranking function at time t' should be $\varrho_{\varrho \to \varrho'}(\cdot) : \mathcal{A} \to N \cup \{\infty\}$,

$$\varrho_{\varrho \to \varrho'}(\cdot) = \min\left\{\varrho(\cdot \mid E_i) + \varrho'(E_i) : i \in I\right\},$$

where $\{E_i \in \mathcal{E} : i \in I\}$ is a set of propositions with $\varrho'(E_i) < \infty$ for all $i \in I$ for which there is no proper superset $\{B_j \in \mathcal{E} : j \in J\}$ of propositions with $\varrho'(E_j) < \infty$ for all $j \in J$.

Spohn revision satisfies the principle of categorical matching (without assuming that propositions with extreme ranks keep their extreme ranks). Furthermore, Spohn revision turns regular ranking functions into regular ranking functions, provided you update by a regular ranking function. The good news is you always can.

Field (1978) (see also Garber 1980) for the probabilistic side, and Shenoy (1991) for the ranktheoretic side propose update rules characterizing the evidence as such, independently of the prior degree of belief function.

Update Rule 6.7 (Field Conditionalization). If $\mathrm{Pr}(\cdot) : \mathcal{A} \to \Re$ is your subjective probability measure at time t and between t and t' your subjective probabilities on the partition $\{E_i \in \mathcal{A} : i \in N\}$ change with strength $\alpha_i \in (-\infty, \infty)$, where $\sum_i \alpha_i = 0$, and your positive subjective probabilities change on no finer partition, then your subjective probability measure at time t' should be $\Pr_{E_i \uparrow \alpha_i}(\cdot) : \mathcal{A} \to \Re$,

$$\begin{aligned} \Pr{}_{E_i \uparrow \alpha_i}(\cdot) &= \frac{\sum_i e^{\alpha_i} \cdot \Pr(\cdot \cap E_i)}{\sum_i e^{\alpha_i} \cdot \Pr(E_i)} \\ &= \sum{}_i \Pr(\cdot \cap E_i) \cdot \frac{e^{\alpha_i}}{s}, \quad s = \sum{}_i e^{\alpha_i} \cdot \Pr(E_i) \\ &= \sum{}_i \Pr(\cdot \mid E_i) \cdot q_i, \quad q_i = \frac{e^{\alpha_i} \cdot \Pr(E_i)}{s}, \quad \sum{}_i q_i = 1 \\ &= \Pr{}_{E_i \to q_i}(\cdot). \end{aligned}$$

Field conditionalization leads from one probability measure to another probability measure. Furthermore, Field conditionalization leads from one regular probability measure to another regular probability measure. Both these things would be different if the α_i were allowed to equal ∞. As Field (1978: 363) observes, the α_i cannot be the values of a probability measure. This is the reason why there is no Field revision.

UPDATE RULE 6.8 (Shenoy Conditionalization). If $\varrho(\cdot) : \mathcal{A} \to N \cup \{\infty\}$ is your ranking function at time t and between t and t' your ranks on the partition $\{E_i \in \mathcal{A} : i \in I\}$ change with strength $z_i \in N$, where $\min\{z_i : i \in I\} = 0$, and your finite ranks change on no finer partition, then your ranking function at time t' should be $\varrho_{E_i \uparrow z_i}(\cdot) : \mathcal{A} \to N \cup \{\infty\}$,

$$\varrho_{E_i \uparrow z_i}(\cdot) = \min\{\varrho(\cdot \cap E_i) + z_i - m\}, \quad m = \min\{z_i + \varrho(E_i) : i \in N\}.$$

Shenoy conditionalization leads from one ranking function to another ranking function, and from one regular ranking function to another regular ranking function. These claims depend on the assumption that the z_i are finite. As an aside, note that plain conditionalization results as a limiting ($z_i \to \infty$) case of Shenoy conditionalization, whereas it is a special ($z_i = \infty$) case of Spohn conditionalization. The same is true for strict conditionalization in relation to Field and Jeffrey conditionalization, respectively (Field 1978: 365).

Shenoy conditionalization is evidence oriented in the sense that $\varrho_{E_i \uparrow z_i}(E_i) - \varrho(E_i) = z_i - m$. Note that there is no loss of generality in restricting the parameters z_i to N rather than the set of integers Z. A change in the rank of $E \in \mathcal{A}$ with strength $-z$, $z \in N$, is nothing but a change in the rank of $\overline{E} \in \mathcal{A}$ with strength z. More importantly, as Shenoy (1991: 173) observes, in contrast to the probabilistic case, where the α_i cannot be described as the values of a probability measure, the z_i now can be described as the values of a ranking function. Hence there also is

Update Rule 6.9 (Shenoy Revision). If $\varrho(\cdot): \mathcal{A} \to N \cup \{\infty\}$ is your ranking function at time t and between t and t' your ranking function on the field $\mathcal{E} \subseteq \mathcal{A}$ changes by $\varrho'(\cdot): \mathcal{E} \to N \cup \{\infty\}$, and the finite part of your ranking function changes on no field $\mathcal{B}$ with $\mathcal{E} \subset \mathcal{B} \subseteq \mathcal{A}$, then your ranking function at time t' should be $\varrho_{\varrho \uparrow \varrho'}(\cdot): \mathcal{A} \to N \cup \{\infty\}$,

$$\varrho_{\varrho \uparrow \varrho'}(\cdot) = \min{}_i \left\{ \varrho(\cdot \cap E_i) + \varrho'(E_i) - m \right\}, \quad m = \min{}_i \left\{ \varrho'(E_i) + \varrho(E_i) \right\},$$

where $\{E_i \in \mathcal{E} : i \in I\}$ is a set of exclusive propositions with $\varrho'(E_i) < \infty$ for all $i \in I$ for which there is no superset $\{B_j \in \mathcal{E} : j \in J\}$ of propositions with $\varrho'(E_j) < \infty$ for all $j \in J$.

Shenoy revision satisfies the principle of categorical matching and leads from one ranking function to another. However, in order to assure this we have to make some cosmetic adjustments for the case of $m = \infty$, or ruling it out by fiat. Alternatively, we could assume the new ranking function to be regular. In any case, Shenoy revision with two regular ranking functions gives rise to another regular ranking function.

Just as Jeffrey and Field conditionalization are interdefinable, so are Spohn and Shenoy revision (and hence conditionalization):

$$\varrho_{\varrho \to \varrho'}(\cdot) = \varrho_{\varrho \uparrow \varrho''}(\cdot), \quad \varrho''(\cdot) = \varrho'(\cdot) - \varrho(\cdot).$$

There is the Lewis-Teller Dutch Book Argument for strict conditionalization (Teller 1973), and there is a Dutch Book Argument for Jeffrey conditionalization (Armendt 1980). What about a Consistency Argument for plain and Spohn conditionalization as well as Spohn and Shenoy revision?

7. Consistency Theorems for Conditionalization and Revision

An agent's conditional degree of entrenchment for a proposition A conditional on a proposition E is defined as the number of independent and mp-reliable information sources providing the conditional information A given E that it takes for the agent to give up her conditional disbelief that A given E. If the agent does not conditionally disbelieve A given E to begin with, it does not take any conditional information source providing the conditional information A given E to make her stop conditionally disbelieving A given E.

In this case her conditional degree of entrenchment for A given E is 0. If no finite number of conditional information sources is able to make the agent stop conditionally disbelieving A given E, her conditional degree of entrenchment for A given E is ∞. To receive the conditional information A given E is, among others, to receive the conditional information B given E, for any proposition $B \supseteq A$. To independently and mp-reliably receive n times the conditional information A given E is, among others, to independently and mp-reliably receive n times the conditional information B given E. It is *not* to independently and mp-reliably receive m times the conditional information B given E, for some $m \neq n$. If you tell me that the temperature today at noon will be 93° Fahrenheit provided it does not rain, you also tell me that the temperature today at noon will be between 90° and 96° Fahrenheit provided it does not rain. The reliability with which I get the second conditional information is *exactly* the same as the reliability with which I get the first conditional information (it is still *you* who tells me so). The difference between the two is a difference in conditional informational content.

THEOREM 7.1 (Consistency Theorem for Plain Conditionalization). *Let $\mathcal{A}$ be a field of propositions over the set of possibilities W. Suppose evidence comes in form of a proposition only, and an agent with a finitely / countably / completely minimitive ranking function ϱ on $\mathcal{A}$ becomes certain of $E \in \mathcal{A}$ in the sense that her new entrenchment function ϱ' on $\mathcal{A}$ is such that $\varrho'(E) = 0$ and $\varrho'\left(\overline{E}\right) = \infty$, and there is no logically stronger proposition with this property. Then the following are equivalent:*

1. *The agent updates ϱ according to plain conditionalization.*
2. *Every possible current or future belief set (unconditional or conditional on E) based on ϱ' is consistent and deductively closed in the finite / countable / complete sense.*
3. *The agent's degrees of entrenchment conditional on E remain unaffected, and ϱ' is a finitely / countably / completely minimitive ranking function.*

Conditional degrees of entrenchment are assumed to be numbers from $N \cup \{\infty\}$.

THEOREM 7.2 (Consistency Theorem for Spohn Conditionalization). *Let $\mathcal{A}$ be a field of propositions over the set of possibilities W. Suppose evidence comes in form of new degrees of entrenchment for the propositions of a partition of W, and an agent with a finitely / countably / completely minimitive ranking function ϱ on $\mathcal{A}$ changes her degrees of entrenchment on $\{E_i \in \mathcal{A} : i \in I\}$ to $n_i \in N \cup \{\infty\}$ with $\min_i\{n_i\} = 0$ ($n_i = \infty$ for $E_i = \emptyset$*

and $n_i = 0$ for $E_i = W$) in the sense that her new entrenchment function ϱ' on $\mathcal{A}$ is such that $\varrho'(E_i) = n_i$, and her finite ranks change on no finer partition. Then the following are equivalent:

1. *The agent updates ϱ according to Spohn conditionalization.*
2. *Every possible current or future belief set (unconditional or conditional on some E_i) based on ϱ' is consistent and deductively closed in the finite / countable / complete sense.*
3. *The agent's degrees of entrenchment conditional on each E_i remain unaffected, and ϱ' is a finitely / countably / completely minimitive ranking function.*

Conditional degrees of entrenchment are assumed to be numbers from $N \cup \{\infty\}$.

THEOREM 7.3 (Consistency Theorem for Spohn Revision). *Let $\mathcal{A}$ be a field of propositions over the set of possibilities W. Suppose evidence comes in form of a new entrenchment function on a subfield $\mathcal{E}$ of $\mathcal{A}$, and an agent with a finitely / countably / completely minimitive ranking function ϱ on $\mathcal{A}$ adopts $\varrho^* : \mathcal{E} \to N \cup \{\infty\}$ as her new entrenchment function on $\mathcal{E}$ in the sense that her new entrenchment function ϱ' on $\mathcal{A}$ coincides with ϱ^* on $\mathcal{E}$, and the finite part of ϱ does not change on a field $\mathcal{B}$ with $\mathcal{E} \subset \mathcal{B} \subseteq \mathcal{A}$. Let $\{E_i \in \mathcal{E} : i \in I\}$ be a set of propositions with $\varrho^*(E_i) < \infty$ for all $i \in I$ for which there is no proper superset $\{E_j \in \mathcal{E} : j \in J\}$ of propositions with $\varrho^*(E_j) < \infty$ for all $j \in J$. Then the following are equivalent:*

1. *The agent updates ϱ according to Spohn revision.*
2. *Every possible current or future belief set (unconditional or conditional on some E_i) based on ϱ' is consistent and deductively closed in the finite / countable / complete sense.*
3. *The agent's degrees of entrenchment conditional on each E_i remain unaffected, and ϱ' is a finitely / countably / completely minimitive ranking function.*

Conditional degrees of entrenchment are assumed to be numbers from $N \cup \{\infty\}$.

PROOF. $1 \Rightarrow 2$: If the agent updates ϱ according to Spohn revision, her entrenchment function after adopting ϱ^* on $\mathcal{E}$ is $\varrho_{\varrho \to \varrho^*}$. $\varrho_{\varrho \to \varrho^*}$ is a finitely / countably / completely minimitive ranking function if ϱ is. So each current belief set (unconditional or conditional on some E_i) based on $\varrho_{\varrho \to \varrho^*}$ is consistent and deductively closed in the finite / countable / complete sense. Moreover, the same is true for every possible future belief set (unconditional or conditional on some E_i). For, by assumption, evidence comes in form of a new entrenchment function on a subfield of the original field, and Spohn revision leads from one finitely / countably / completely minimitive ranking function to another.

$2 \Rightarrow 3$: Suppose the agent adopts ϱ' as her entrenchment function on $\mathcal{A}$ after adopting ϱ^* on $\mathcal{E}$. If ϱ' is no ranking function, we proceed as in the proof of theorem 4.2.

By assumption, there is no proper superfield of $\mathcal{E}$ on which the positive part of ϱ changes between t and t'. So for any $A \in \mathcal{A}$ and all E_i: if $A \notin Bel_{\varrho(\cdot|E_i)}$ and $A \in Bel_{\varrho'(\cdot|E_i)}$, then $E_i \subseteq A$; and if $A \in Bel_{\varrho(\cdot|E_i)}$ and $A \notin Bel_{\varrho'(\cdot|E_i)}$, then $E_i \subseteq \overline{A}$.

Now suppose the agent changes her degrees of entrenchment conditional on some E_i. Then $\varrho(A \mid E_i) \neq \varrho'(A \mid E_i)$ for some $A \in \mathcal{A}$.

(1) Suppose first $\varrho(A \mid E_i) < \varrho'(A \mid E_i)$.

(1.1) If $\varrho(A \mid E_i) = 0$, i.e. $\overline{A} \notin Bel_{\varrho(\cdot|E_i)}$, then $\varrho'(A \mid E_i) > 0$, i.e. $\overline{A} \in Bel_{\varrho'(\cdot|E_i)}$. But then $E_i \subseteq \overline{A}$, and so $Bel_{\varrho(\cdot|E_i)}$ is not deductively closed in the finite, countable, or complete sense – contradicting the assumption that ϱ is a ranking function.

(1.2) If $\varrho(A \mid E_i) = n > 0$, we have the following situation:

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0 $\quad \varrho(A \mid E_i) = n \quad \varrho'(A \mid E_i) = n' > n$

Assume the agent receives evidence equivalent to being given the conditional information A given E_i by at least n, but fewer than n' independent and mp-reliable information sources. The resulting conditional entrenchment functions $\varrho_{n^*}(\cdot \mid E_i)$ and $\varrho'_{n^*}(\cdot \mid E_i)$ after independently and mp-reliably receiving $n^* \in [n, n')$ times the conditional information A given E_i are such that $\varrho_{n^*}(A \mid E_i) = \varrho(A \mid E_i) - n^* = 0$ and $\varrho'_{n^*}(A \mid E_i) = \varrho'(A \mid E_i) - n^* > 0$. Now we have the following situation:

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$\varrho_{n^*}(A \mid E_i) \leq 0 \quad \varrho'_{n^*}(A \mid E_i) > 0$

Hence $\overline{A} \notin Bel_{\varrho_{n^*}(\cdot|E_i)}$ and $\overline{A} \in Bel_{\varrho'_{n^*}(\cdot|E_i)}$, and so $E_i \subseteq \overline{A}$. Therefore $Bel_{\varrho_{n^*}(\cdot|E_i)}$ is not deductively closed in the finite, countable, or complete sense – contradicting the assumption that ϱ is a ranking function.

(2) Now suppose $\varrho(A \mid E_i) > \varrho'(A \mid E_i)$.

(2.1) If $\varrho'(A \mid E_i) = 0$, i.e. $\overline{A} \notin Bel_{\varrho'(\cdot|E_i)}$, then $\varrho(A \mid E_i) > 0$, i.e. $\overline{A} \in Bel_{\varrho(\cdot|E_i)}$. But then $E_i \subseteq A$, and so $Bel_{\varrho(\cdot|E_i)}$ is inconsistent in the finite, countable, and complete sense if it is deductively closed in the finite, countable, or complete sense – contradicting the assumption that ϱ is a ranking function.

(2.2) If $\varrho'(A \mid E_i) = n > 0$, we have the following situation:

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0 $\quad \varrho'(A \mid E_i) = n \quad \varrho(A \mid E_i) = n' > n$

Assume the agent receives evidence equivalent to being given the conditional information A given E_i by at least n, but fewer than n' independent and mp-reliable information sources. The resulting conditional entrenchment functions $\varrho_{n^*}(\cdot \mid E_i)$ and $\varrho'_{n^*}(\cdot \mid E_i)$ after independently and mp-reliably receiving $n^* \in [n, n')$ times the conditional information A given E_i are such that $\varrho_{n^*}(A \mid E_i) = \varrho(A \mid E_i) - n^* > 0$ and $\varrho'_{n^*}(A \mid E_i) = \varrho'(A \mid E_i) - n^* = 0$. Now we have the following situation:

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$\varrho'_{n^*}(A \mid E_i) \leq 0 \quad \varrho_{n*}(A \mid E_i) > 0$

Hence $\overline{A} \notin Bel_{\varrho'_{n^*}(\cdot \mid E_i)}$ and $\overline{A} \in Bel_{\varrho_{n^*}(\cdot \mid E_i)}$, and therefore $E_i \subseteq A$. But then $Bel_{\varrho_{n^*}(\cdot \mid E_i)}$ is inconsistent in the finite, countable, and complete sense if it is deductively closed in the finite, countable, or complete sense – contradicting the assumption that ϱ is a ranking function.

$3 \Rightarrow 1$: Suppose the agent adopts the finitely / countably / completely minimitive ϱ' as her ranking function on $\mathcal{A}$ after adopting ϱ^* on $\mathcal{E}$. By assumption we have for all E_i: $\varrho(\cdot \mid E_i) = \varrho'(\cdot \mid E_i)$. As $\varrho(\cdot \mid E_i) = \varrho_{\varrho \to \varrho^*}(\cdot \mid E_i)$ and $\varrho_{\varrho \to \varrho^*}(E_i) = \varrho'(E_i)$, this entails $\varrho_{\varrho \to \varrho^*}(A) = \varrho'(A)$ for all $A \in \mathcal{A}$. ■

Theorem 7.4 (Consistency Theorem for Shenoy Conditionalization). *Let $\mathcal{A}$ be a field of propositions over the set of possibilities W. Suppose evidence comes in form of new degrees of entrenchment for the propositions of a partition of W, and an agent with a finitely / countably / completely minimitive ranking function ϱ on $\mathcal{A}$ changes her degrees of entrenchment on $\{E_i \in \mathcal{A} : i \in I\}$ by $z_i \in N$, where $\min\{z_i : i \in I\} = 0$, in the sense that her new entrenchment function ϱ' on $\mathcal{A}$ is such that $\varrho'(E_i) = \varrho(E_i) + z_i - m$, $m = \min\{z_i + \varrho(E_i) : i \in N\}$, and her finite ranks change on no finer partition. Then the following are equivalent:*

1. *The agent updates ϱ according to Shenoy conditionalization.*
2. *Every possible current or future belief set (unconditional or conditional on some E_i) based on ϱ' is consistent and deductively closed in the finite / countable / complete sense.*
3. *The agent's degrees of entrenchment conditional on each E_i remain unaffected, and ϱ' is a finitely / countably / completely minimitive ranking function.*

Conditional degrees of entrenchment are assumed to be numbers from $N \cup \{\infty\}$.

THEOREM 7.5 (Consistency Theorem for Shenoy Revision). *Let* $\mathcal{A}$ *be a field of propositions over the set of possibilities* W. *Suppose evidence comes in form of an additional entrenchment function on a subfield* $\mathcal{E}$ *of* $\mathcal{A}$, *and an agent with a finitely / countably / completely minimitive ranking function* ϱ *on* $\mathcal{A}$ *revises* ϱ *by the finitely / countably / completely minimitive ranking function* $\varrho^* : \mathcal{E} \to N \cup \{\infty\}$ *in the sense that her new entrenchment function* ϱ' *on* $\mathcal{A}$ *coincides with* $\varrho_{\varrho \uparrow \varrho^*}$ *on* $\mathcal{E}$, *and the finite part of* ϱ *does not change on a field* $\mathcal{B}$ *with* $\mathcal{E} \subset \mathcal{B} \subseteq \mathcal{A}$. *Let* $\{E_i \in \mathcal{E} : i \in I\}$ *be a set of propositions with* $\varrho^*(E_i) < \infty$ *for all* $i \in I$ *for which there is no proper superset* $\{E_j \in \mathcal{E} : j \in J\}$ *of propositions with* $\varrho^*(E_j) < \infty$ *for all* $j \in J$. *Then the following are equivalent:*

1. *The agent updates* ϱ *according to Shenoy revision.*
2. *Every possible current or future belief set (unconditional or conditional on some* E_i*) based on* ϱ' *is consistent and deductively closed in the finite / countable / complete sense.*
3. *The agent's degrees of entrenchment conditional on each* E_i *remain unaffected, and* ϱ' *is a finitely / countably / completely minimitive ranking function.*

Conditional degrees of entrenchment are assumed to be numbers from $N \cup \{\infty\}$.

PROOF. $1 \Rightarrow 2$: If the agent updates ϱ according to Shenoy revision, her entrenchment function after revision by ϱ^* is $\varrho_{\varrho \uparrow \varrho^*}$. Subject to cosmetic adjustments, $\varrho_{\varrho \uparrow \varrho^*}$ is a finitely / countably / completely minimitive ranking function if ϱ is. So each current belief set (unconditional or unconditional on some E_i) based on $\varrho_{\varrho \uparrow \varrho^*}$ is consistent and deductively closed in the finite / countable / complete sense. Moreover, the same is true for every possible future belief set (unconditional or conditional on some E_i). For, by assumption, evidence comes in form of an additional entrenchment function on some subfield of the original field of propositions, and, subject to cosmetic adjustments, Shenoy revision leads from one finitely / countably / completely minimitive ranking function to another.
$2 \Rightarrow 3$: Suppose the agent adopts ϱ' as her entrenchment function after revision by ϱ^* on $\mathcal{E}$. If ϱ' is no ranking function, we proceed as in the proof of theorem 4.2.

By assumption, there is no proper superfield of $\mathcal{E}$ on which the positive part of ϱ changes between t and t'. So for any $A \in \mathcal{A}$ and all E_i: if $A \notin Bel_{\varrho(\cdot|E_i)}$ and $A \in Bel_{\varrho'(\cdot|E_i)}$, then $E_i \subseteq A$; and if $A \in Bel_{\varrho(\cdot|E_i)}$ and $A \notin Bel_{\varrho'(\cdot|E_i)}$, then $E_i \subseteq \overline{A}$.

Now suppose the agent changes her degrees of entrenchment conditional on some E_i. Then $\varrho(A \mid E_i) \neq \varrho'(A \mid E_i)$ for some $A \in \mathcal{A}$.

(1) Suppose first $\varrho(A \mid E_i) < \varrho'(A \mid E_i)$.

(1.1) If $\varrho(A \mid E_i) = 0$, i.e. $\overline{A} \notin Bel_{\varrho(\cdot|E_i)}$, then $\varrho'(A \mid E_i) > 0$, i.e. $\overline{A} \in Bel_{\varrho'(\cdot|E_i)}$. But then $E_i \subseteq \overline{A}$, and so $Bel_{\varrho(\cdot|E_i)}$ is not deductively closed in the finite, countable, or complete sense – contradicting the assumption that ϱ is a ranking function.

(1.2) If $\varrho(A \mid E_i) = n > 0$, we have the following situation:

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0 $\quad \varrho(A \mid E_i) = n \quad \varrho'(A \mid E_i) = n' > n$

Assume the agent receives evidence equivalent to being given the conditional information A given E_i by at least n, but fewer than n' independent and mp-reliable information sources. The resulting conditional entrenchment functions $\varrho_{n^*}(\cdot \mid E_i)$ and $\varrho'_{n^*}(\cdot \mid E_i)$ after independently and mp-reliably receiving $n^* \in [n, n')$ times the conditional information A given E_i are such that $\varrho_{n^*}(A \mid E_i) = \varrho(A \mid E_i) - n^* = 0$ and $\varrho'_{n^*}(A \mid E_i) = \varrho'(A \mid E_i) - n^* > 0$. Now we have the following situation:

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$\varrho_{n^*}(A \mid E_i) \leq 0 \quad \varrho'_{n^*}(A \mid E_i) > 0$

Hence $\overline{A} \notin Bel_{\varrho_{n^*}(\cdot|E_i)}$ and $\overline{A} \in Bel_{\varrho'_{n^*}(\cdot|E_i)}$, and so $E_i \subseteq \overline{A}$. Therefore $Bel_{\varrho_{n^*}(\cdot|E_i)}$ is not deductively closed in the finite, countable, or complete sense – contradicting the assumption that ϱ is a ranking function.

(2) Now suppose $\varrho(A \mid E_i) > \varrho'(A \mid E_i)$.

(2.1) If $\varrho'(A \mid E_i) = 0$, i.e. $\overline{A} \notin Bel_{\varrho'(\cdot|E_i)}$, then $\varrho(A \mid E_i) > 0$, i.e. $\overline{A} \in Bel_{\varrho(\cdot|E_i)}$. But then $E_i \subseteq A$, and so $Bel_{\varrho(\cdot|E_i)}$ is inconsistent in the finite, countable, and complete sense if it is deductively closed in the finite, countable, or complete sense – contradicting the assumption that ϱ is a ranking function.

(2.2) If $\varrho'(A \mid E_i) = n > 0$, we have the following situation:

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0 $\quad \varrho'(A \mid E_i) = n \quad \varrho(A \mid E_i) = n' > n$

Assume the agent receives evidence equivalent to being given the conditional information A given E_i by at least n, but fewer than n' independent and mp-reliable information sources. The resulting conditional entrenchment functions $\varrho_{n^*}(\cdot \mid E_i)$ and $\varrho'_{n^*}(\cdot \mid E_i)$ after independently and mp-reliably receiving $n^* \in [n, n')$ times the conditional information A given E_i are such that $\varrho_{n^*}(A \mid E_i) = \varrho(A \mid E_i) - n^* > 0$ and $\varrho'_{n^*}(A \mid E_i) = \varrho'(A \mid E_i) - n^* = 0$. Now we have the following situation:

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$\varrho'_{n^*}(A \mid E_i) \leq 0$ $\quad\quad$ $\varrho_{n^*}(A \mid E_i) > 0$

Hence $\overline{A} \notin Bel_{\varrho'_{n^*}(\cdot|E_i)}$ and $\overline{A} \in Bel_{\varrho_{n^*}(\cdot|E_i)}$, and therefore $E_i \subseteq A$. But then $Bel_{\varrho_{n^*}(\cdot|E_i)}$ is inconsistent in the finite, countable, and complete sense if it is deductively closed in the finite, countable, or complete sense – contradicting the assumption that ϱ is a ranking function.

3 $\Rightarrow$ 1: Suppose the agent adopts the finitely / countably / completely minimitive ϱ' as her ranking function on $\mathcal{A}$ after revision by ϱ^* on $\mathcal{E}$. By assumption we have for all E_i: $\varrho(\cdot \mid E_i) = \varrho'(\cdot \mid E_i)$. As $\varrho(\cdot \mid E_i) = \varrho_{\varrho\uparrow\varrho^*}(\cdot \mid E_i)$ and $\varrho_{\varrho\uparrow\varrho^*}(E_i) = \varrho(E_i) + \varrho^*(E_i) - \min\{\varrho(E_i) + \varrho^*(E_i) : i \in I\} = \varrho'(E_i)$, this entails $\varrho_{\varrho\uparrow\varrho^*}(A) = \varrho'(A)$. ■

These theorems allow us to drop the temporary *assumption* of theorem 4.2 that updating leads from one ranking function to another. Together the theorems of section 4 and 7 show that the package consisting of the axioms and the update rules of the ranking calculus are nothing but a diachronic version of consistency and deductive closure.

8. Conclusion

According to the Consistency Argument, to violate the ranking axioms is to possibly have beliefs that are not both consistent and deductively closed. Most likely it is not epistemically defective to have such beliefs. What is more likely to be epistemically defective is to knowingly have such beliefs. The ranking axioms assume the agent to be logically omniscient on the object level. Among others, this assumption shows up by taking propositions rather than sentences or other representations to be the objects of belief. If we extend this assumption to the meta level so that it includes propositions about one's own epistemic state[5], the distinction between inconsistency and known inconsistency breaks down: to have beliefs that are inconsistent or not deductively closed is to knowingly have such beliefs. Given a link between degrees of disbelief and degrees of entrenchment, the normative force of the Consistency Argument is then proportional to how odd one takes

[5]I am grateful to Eric Swanson for pointing out to me that there are two levels with respect to which the agent can be logically omniscient, and that the distinction between inconsistency and known inconsistency breaks down only if we assume the agent to be logically omniscient on the meta level.

the possibility of knowingly having beliefs that are not both consistent and deductively closed.

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