Milne's Argument for the Log-Ratio Measure*

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This article shows that a slight variation of the argument in Milne 1996 yields the log-likelihood ratio l rather than the log-ratio measure r as "the one true measure of confirmation."

1. Introduction. Peter Milne (1996) shows that

$$r(H, E, B) = \log[\Pr(H|E \cap B)/\Pr(H|B)]$$

is "the one true measure of confirmation" in the sense that r is the one and only function satisfying the following five constraints on measures of confirmation C.

1. $C(H, E, B) \gtreqless 0$ iff $\Pr(H|E \cap B) \gtreqless \Pr(H|B)$.
2. $C(H, E, B)$ is a function that the values $\Pr(X|B)$ and $\Pr(Y|Z \cap B)$ assume on the at most sixteen truth-functional combinations X, Y, Z of E and H.

3a. If $\Pr(E|H \cap B) < \Pr(F|H \cap B)$ and $\Pr(E|B) = \Pr(F|B)$, then $C(H, E, B) \geq C(H, F, B)$.

3b. If $\Pr(E|H \cap B) = \Pr(F|H \cap B)$ and $\Pr(E|B) < \Pr(F|B)$, then $C(H, E, B) \geq C(H, F, B)$.

4a. $C(H, E \cap F, B) - C(H, E \cap G, B)$ is determined by $C(H, E, B)$ and the difference $C(H, F, E \cap B) - C(H, G, E \cap B)$.

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4b. If $C(H, E \cap F, B) = 0$, then $C(H, E, B) + C(H, F, E \cap B) = 0$.
5. If $\Pr(E|H \cap B) = \Pr(E|T \cap B)$, then $C(H, E, B) = C(T, E, B)$.

Among these constraints, 1, 3, and 5 concern the relation between confirmation and probability, while 2 and 4 concern confirmation alone. I will only be concerned with the former.

Constraint 1 is logically equivalent to

1^+. $C(H, E, B) \gtreqless 0$ iff $\Pr(E \mid H \cap B) \gtreqless \Pr(E|B)$.

This makes clear that 1, 3, and 5 say what happens to confirmation $C(H, E, B)$ if various relations between the likelihood of hypothesis H on evidence E and background information B, $\Pr(E|H \cap B)$, and the prior of E given B, $\Pr(E|B)$, obtain.

Constraint 1^+ is logically equivalent to

1*. $C(H, E, B) \gtreqless 0$ iff $\Pr(E|H \cap B) \gtreqless \Pr(E|\bar{H} \cap B)$.

Similarly, Constraint 3b is logically equivalent to 3b*: if $\Pr(E|H \cap B) = \Pr(F|H \cap B)$ and $\Pr(E|\bar{H} \cap B) < \Pr(F|\bar{H} \cap B)$, then $C(H, E, B) \geq C(H, F, B)$. While Constraints 1^+ and 3b focus on relations between likelihoods and priors, Constraints 1* and 3b* say the same thing by focusing on relations between likelihoods and what, following Fitelson (2007), we call *catch-alls*: $\Pr(E|\bar{H} \cap B)$. Let us see where this shift in focus takes us.

Regarding Constraint 3a Milne (1996, 21) states that it "corresponds more or less to the claim . . . that, other things being equal, a theory is better confirmed by evidence the more likely the theory makes the evidence." More than one thing can be equal, though. Often not all of them can be equal simultaneously. According to Constraint 3a, the prior of the evidence is held fixed: $\Pr(E|B)$ is equal to $\Pr(F|B)$.

Consider the catch-all counterpart

3a*. If $\Pr(E|H \cap B) < \Pr(F|H \cap B)$ and $\Pr(E|\bar{H} \cap B) = \Pr(F|\bar{H} \cap B)$, then $C(H, E, B) \leq C(H, F, B)$.

According to 3a*, the catch-all, the likelihood of $\bar{H}$ on the evidence, is held fixed: $\Pr(E|\bar{H} \cap B)$ is equal to $\Pr(F|\bar{H} \cap B)$. Given that the theory makes the one evidence more likely than the other, that is, $\Pr(E|H \cap B) < \Pr(F|H \cap B)$, not both of these other things can be equal.

Regarding Constraint 5, Milne says that it "is a weak consequence of the Likelihood Principle" (1996, 22):

> In comparing the evidential bearing (relative to background knowledge B) of E on the hypotheses H and T we need consider only $\Pr(E|H \cap B)$ and $\Pr(E|T \cap B)$.

Note that, in the presence of Constraints 1–4, Constraint 5 is equivalent to the otherwise stronger

5^+. If $\Pr(E|H \cap B) = \Pr(F|T \cap B)$ and $\Pr(E|B) = \Pr(F|B)$, then $C(H, E, B) = C(T, F, B)$.

This is so because $r(H, E, B)$ satisfies 5^+.

Here is the catch-all counterpart of 5^+:

5*. If $\Pr(E|H \cap B) = \Pr(F|T \cap B)$ and $\Pr(E|\bar{H} \cap B) = \Pr(F|\bar{T} \cap B)$, then $C(H, E, B) = C(T, F, B)$.

Let us rename Constraints 2 and 4 by 2* and 4*, respectively. Then things can be put as follows. In the presence of Constraints 2 and 4, the conjunction of Constraints 1, 3, and 5 says that $C(H, E, B)$ is a function of the likelihood of H on E, $\Pr(E|H \cap B)$, and the prior of E, $\Pr(E|B)$—increasing with the former, and decreasing with the latter.

In the presence of 2* and 4*, the conjunction of 1*, 3*, and 5* says that $C(H, E, B)$ is a function of the likelihood of H on E, $\Pr(E|H \cap B)$, and the catch-all, that is, the likelihood of $\bar{H}$ on E, $\Pr(E|\bar{H} \cap B)$—increasing with the former, and decreasing with the latter.

2. Catch-Alls or Priors? A variation variation of Milne's proof (presented in Appendix 1) shows that

$$l(H, E, B) = \log[\Pr(E|H \cap B)/\Pr(E|\bar{H} \cap B)]$$

is another true measure of confirmation in the sense that l is the one and only function satisfying 1*–5*.

As Fitelson (2001, 29) observes, l satisfies 1–4. It is worth noting that r satisfies 1*–4*. So the difference between r and l lies in 5 versus 5*: l does not satisfy 5, and r does not satisfy 5*.

Thus r and l agree that confirmation depends on the likelihood of H on E, $\Pr(E|H \cap B)$, and one other factor. They also agree on how to compare the likelihood of H on E to the other factor, namely, by taking logarithms of ratios. What they disagree about is the other factor the likelihoods of H on E should be compared to: r says the other factor is the prior of the evidence E, $\Pr(E|B)$, while l says it is the catch-all, that is, the likelihood of $\bar{H}$ on the evidence E, $\Pr(E|\bar{H} \cap B)$.

3. Odds or Probabilities? Things can be put differently still. Let $O(H|B)$ and $O(H|E \cap B)$ stand for the prior and posterior odds of H, respectively,

$$O(H|B) = \frac{\Pr(H|B)}{\Pr(\bar{H}|B)} \text{ and } O(H|E \cap B) = \frac{\Pr(H|E \cap B)}{\Pr(\bar{H}|E \cap B)}.$$

Then, as Joyce (2003, table 5) observes,

$$r(H, E, B) = \log\left[\frac{\Pr(H|E \cap B)}{\Pr(H|B)}\right] \text{ and } l(H, E, B) = \log\left[\frac{O(H|E \cap B)}{O(H|B)}\right].$$

Seen this way r and l agree that it is differences between priors and posteriors that matter for confirmation. They also agree on how to measure those differences, viz., by taking the logarithm of the ratio of posterior over prior. What they disagree about is, to speak with Joyce (2003, Section 3), the question whether we should consider differences in "total evidence" as measured by $\Pr(H|E \cap B)$ and $\Pr(H|B)$, or differences in "net evidence" as measured by $O(H|E \cap B)$ and $O(H|B)$.

4. Conclusion. Milne (1996) presents his argument as a desideratum/explicatum argument for r as opposed to other measures of confirmation. His confirmation theoretic monism presupposes that there is one and only one true measure of confirmation. Joyce (2003, Section 3), on the other hand, favors a confirmation theoretic pluralism according to which, among others, each of r and l "measures an important evidential relationship, but that the relationships they measure are importantly different."[1]

This pluralistic view suggests to view Milne's (1996) argument and the above variation not so much as arguments for or against one particular measure of confirmation. Rather, they can be viewed as *characterizations* that tell us, descriptively, what particular measures focus on, without telling us, prescriptively, what we *should* focus on. The latter, normative question seems to be beyond the reach of desiderata/explicata approaches, but to belong to the realm of means-ends epistemology or epistemic consequentialism (Percival 2002; Stalnaker 2002) as exemplified, for probability, by Joyce (1998), and for confirmation, by Huber (2005).

Appendix 1: A Variation of Milne's (1996) Proof

The following proof is entirely due to Milne 1996, Appendix 1, although all errors are, of course, mine.

Constraint 2* entails that $C(H, E, B)$ is a function of $\Pr(E|H \cap B)$, $\Pr(E|\bar{H} \cap B)$, and $\Pr(H|B)$. Constraint 5* entails that $C(H, E, B)$ is independent of $\Pr(H|B)$. So $C(H, E, B) = F(\Pr(E|H \cap B), \Pr(E|\bar{H} \cap B))$ for some $F{:}[0, 1]^2 \to \Re^*$, where $\Re^* = \Re \cup \{\pm\infty\}$.

1. Actually Joyce (2003) considers e^r and e^l, that is, r and l without the log.

Constraint 1* entails that $F(x, x) = 0$ for all $x \in [0, 1]$. As

$$\Pr(E \cap F | H \cap B) = \Pr(E | H \cap B) \cdot \Pr(F | E \cap H \cap B)$$

$$\Pr(E \cap F | \bar{H} \cap B) = \Pr(E | \bar{H} \cap B) \cdot \Pr(F | E \cap \bar{H} \cap B),$$

Constraint 4* entails that there is a possibly partial $G: \Re^{*2} \to \Re^*$ such that for all $x, y, z_1, z_2, w_1, w_2 \in [0, 1]$

$$F(x \cdot z_1, y \cdot w_1) - F(x \cdot z_2, y \cdot w_2) =$$
$$G(F(x, y), F(z_1, w_1) - F(z_2, w_2)). \quad (1)$$

The range of F is assumed to be a real interval. $F(1, 1) = 0$, and so

$$F(x \cdot z, y \cdot w) - F(x, y) = G(F(x, y), F(z, w)), \quad (2)$$

which yields $G(0, u) = u$ and $G(u, 0) = 0$ for $x = y = 1$ and $z = w = 1$, respectively. Equation (2) and the previous equation give us

$$F(x \cdot z, x \cdot w) = F(x, x) + G(F(x, x), F(z, w)) = F(z, w).$$

If $x/z = y/w$, then $F(x, z) = F((z/w) \cdot y, (x/y) \cdot w)$ and $z/w = x/y$, or $F((w/z) \cdot x, (y/x) \cdot z) = F(y, w)$ and $w/z = y/x$. Hence $F(x, z) = F(t \cdot y, t \cdot w)$ or $F(t \cdot x, t \cdot z) = F(y, w)$ for some $t \in [0,1]$.

Assume without loss of generality that $F(x, z) = F(t \cdot y, t \cdot w) = F(y, w)$ for $t \in [0,1]$. Then $C(H, E, B) = F(x, z) = F(y, w)$ with $x/z = y/w$, and so $C(H, E, B) = H(\Pr(E | H \cap B) / \Pr(E | \bar{H} \cap B))$ for some $H: \Re_{\geq 0} \to \Re *$. For $z_2 = w_2 = 1$ Equation (1) entails

$$H(x \cdot y) = H(x) + G(H(x), H(y)) = H(y) + G(H(y), H(x)). \quad (3)$$

This and Equation (1) give us

$$G(H(x), H(y)) - G(H(x), H(z)) = H(x \cdot y) - H(x \cdot z) \quad (4)$$

$$= G(H(x), H(y) - H(z)), \quad (5)$$

which yields

$$G(t, u + v) = G(t, u) + G(t, v).$$

For integers m, n and $u \cdot m/n$ in the range of F so that $(t, u \cdot m/n)$ is in the domain of G, we thus have $G(t, u \cdot m/n) = (m/n) \cdot G(t, u)$. Constraint 3a* entails that $G(t, u) \leq G(t, v)$ if $u \leq v$. So for all reals r with $u \cdot r$ in the range of F so that $(t, u \cdot r)$ is in the domain of G, $G(t, u \cdot r) = r \cdot G(t, u)$. Hence $G(t, u) = u \cdot g(t)$ for some $g: \Re^* \to \Re_{\geq 0}$ (at this point Milne refers to Aczél 1966, 31–34).

Equation (3) entails

$$H(x \cdot y) - H(x \cdot z) = H(y) - H(z) + G(H(y), H(x)) - G(H(z), H(x)),$$

and so Equation (5) gives us

$$g(H(x)) \cdot (H(y) - H(z)) = \\ H(y) - H(z) + H(x) \cdot (g(H(y)) - g(H(z))). \tag{6}$$

Constraint 1* entails $H(1) = 0$ and that H is not constant, which implies that $g(0) = 1$. For $H(x) \neq 0$ Equation (6) entails

$$g(H(y)) - g(H(z)) = (g(H(x)) - 1) \cdot (H(y) - H(z))/H(x).$$

The left-hand side is independent of x, and so

$$g(H(x) - 1)/H(x) = k$$

for some constant $k \in \Re^*$.

From Equation (3) we have

$$\begin{aligned} H(x \cdot y) &= H(x) + G(H(x), H(y)) \\ &= H(x) + H(y) \cdot g(H(x)) \\ &= H(x) + H(y) \cdot (H(x) \cdot k + 1) \\ &= H(x) + H(y) + k \cdot H(x) \cdot H(y). \end{aligned}$$

Constraint 4b* entails that $H(x) + H(y) = 0$ if $H(x \cdot y) = 0$. $k = 0$, since it is possible that $H(x \cdot y) = 0$ while $H(x) \neq 0$ and $H(y) \neq 0$ (it suffices to consider a case where E is positively relevant for H, F is negatively relevant for H, and $E \cap F$ is independent of H in the sense of some Pr—note that this argument would be problematic if the underlying probability space were fixed). Hence,

$$H(x \cdot y) = H(x) + H(y), \tag{7}$$

and so $H(x^{m/n}) = m/n \cdot H(x)$ for integers m, n. Constrain 3* entails that $H(x) \leq H(y)$ if $x \leq y$, and so $H(x^r) = r \cdot H(x)$ for all $r \in \Re$. (As Milne notes, no assumptions about the domain of H need be made this time, because any number in $\Re^*$ can be the ratio of two probabilities—again, note that this argument would be problematic if the underlying probability space were fixed.) Therefore $H(x) = c \cdot \log x$ for some constant c (at this point Milne refers to Aczél 1966, 39–41) that has to be positive in view of 1* and equals 1 by a suitable choice of the base of log. Hence $C(H, E, B) = \log(\Pr(E|H \cap B)/\Pr(E|\bar{H} \cap B))$.

Appendix 2: Fitelson's (2001) Objection

Fitelson (2001, 28) notes that "Milne's argument *implicitly* requires that the probability function Pr . . . satisfy some rather strong, unmotivated, and unintuitive constraints." In particular, "Milne's argument makes use of certain theorems . . . which force the probability function Pr (and, hence, the spaces over which the measure [of confirmation *C*] is defined) to satisfy various kinds of *continuity* conditions" (Fitelson 2001, 28, note 43). For a discussion of these conditions Fitelson refers to Halpern 1999a, 1999b, where it is shown that Cox's (1946) theorem does not hold in finite domains.

I think it is perfectly reasonable for Milne (and proponents of the above variation of his argument) to require the domain of the measure of confirmation *C* to be infinite. As Halpern (1999a, Section 5; 1999b, Theorem 5) observes, one response is to say that we are not interested in a single domain in isolation, but a notion of belief or confirmation (in his or our case, respectively) that applies uniformly in all domains.

But suppose we are in fact interested in just one single field of propositions $\mathcal{A}$ over which our measure of confirmation *C* is defined. Suppose further $\mathcal{A}$ is finite. Even then the domain of *C* is uncountable, provided we assume *C* does not vary with the underlying probability measure Pr. That is, we only have to think of *C* as a mapping of probability spaces (and not propositions without probabilities) into the reals, and take its domain to be the set of all probability spaces $\langle \mathcal{A}, \mathrm{Pr} \rangle$ (for the fixed $\mathcal{A}$ from above). As far as I can tell, this assumption is implicit in all discussions of incremental confirmation. Rejecting it means to use different measures of confirmation for different probability measures on one fixed domain, rather than uniformly using the same measure of confirmation.

However, the assumption Milne (1996, 24) actually makes is that the *range* of *C* forms a real interval. This implies that the domain of *C* is uncountably infinite. As argued, the latter assumption is reasonable for Milne to make. Obviously it is another question whether the former is, too.

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